INDIAN STATISTICAL INSTITUTE
Bangalore center
Mid-Term Examination
February 24, 2020

Analysis IV B.Math III Instructor : Santhosh Kumar P Total : 30 Marks

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Answer all questions. Each question carries 6 marks.

NOTE: If you use any known result, please state it briefly.

1. Let X be a set and $\mathcal{F}(X, \mathbb{C})$ denote the class of all functions from X to $\mathbb{C}$. Let $\mathcal{V}$ be a vector subspace of $\mathcal{F}(X, \mathbb{C})$ which is complete with respect to some inner product $\langle \cdot, \cdot \rangle$. Suppose that $\mathcal{V} \ni f \mapsto f(x)$ is continuous for each $x \in X$. Then show that $\mathcal{V}$ separates points of X if and only if $d(x,y) = \sup\left\{|f(x) - f(y)| : \ f \in \mathcal{V}, \langle f, f \rangle \leq 1\right\}$ defines a metric on X.

2. Let X be a compact set and $\mathcal{A}$ be a subalgebra of $C(X, \mathbb{R})$. Suppose $f, g \in \overline{\mathcal{A}}$. Prove or disprove that
$$\min\left\{f, \ 2020\,|g|\right\} + \max\left\{\frac{1}{2020}\,|f| + g, \ g\right\} \in \overline{\mathcal{A}}.$$

3. Let $\mathcal{A}$ be a bounded set of differentiable functions from $(0,1)$ to $\mathbb{R}^{1983}$ such that the derivative is given by
$$Df(x) = \begin{bmatrix} x \\ x/2 \\ x/3 \\ \vdots \\ x/1983 \end{bmatrix} f(1/2), \text{ for all } f \in \mathcal{A}.$$
Show that $\mathcal{A}$ is uniformly equicontinuous.

4. (a) State Arzelá-Ascoli theorem and Banach fixed point theorem.

   (b) Let $\mathcal{A}$ be equicontinuous sub family of $C([0,1], \mathbb{R})$. Discuss the possibility that $\mathcal{A}_x := \{f \in \mathcal{A} : \ f(x) = 0\}$ to be compact, for $x \in [0,1]$.

5. Let $A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}_{4\times 4}$ and define $f_A \colon \mathbb{R}^4 \to \mathbb{R}^4$ by $f_A(X) = AX$, for all $X \in \mathbb{R}^4$. Then

(a) Show that f_A is a contraction.

(b) Is $Range(1 - f_A f_{A^t})$ and $Range(1 - f_{A^t} f_A)$ has same dimension. Justify your answer.